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Applications on Multi Dimensional Differential Transform Method for solving Volterra Integral and Integro-Differential Equations

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Abstract— This paper introduces the basic definitions and theorems of two and three-dimensional differential transform method for integral equations. By applying the differential transform method, the integral equations can be transformed to an algebraic equations and solving this equations, we find the approximate solutions of the integral equations.

Index Terms— Integral equations, two dimensional differential transform, three dimensional differential transform, nonlinear volterra integral equation, volterra integro-partial differential equation, homogenuous nonlinear gas dynamic, error analysis,

1 Introduction

Integral equations [10] are encountered in various fields of science and numerous applications (in elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.).

Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations. The integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order. In classifying integro-differential equations, we follow the same category used before according to the limits of integration. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of integral equations. Many other applications in science, statistics and engineering are described by integral equations or integro-differential equations.

The concept of the differential transform method (DTM) was first proposed by Zhou [11]. Integral equations arise in many scientific and engineering problems, this method has many advantages, it efficiently works with different types of linear and nonlinear integral equations and gives an exact solution for some these types of equations without linearization or discretization.

During the last 10 years, significant progress has been made in numerical analysis of one-dimensional version of (1) (see, [3],[4]). However, the numerical methods for two-dimensional integral equation (1) seem to have been discussed in only a few places. Beltyukov [2] proposed a class of explicit Runge-Kutta type methods of order 3 (without analyzing their convergence). Bivariate cubic spline functions method of full continuity was obtained by Singh [8]. Brunner [4] introduced collocation and iterated collocation methods for two-dimensional linear Volterra integral equation. They gave an analysis of global and local convergence properties of collocation method and iterated collocation method, and derived results on attainable orders of global convergence and local superconvergence. Guoqiang and Hayami [5] introduced extrapolation method of iterated collocation solution for two-dimensional nonlinear Volterra integral equation. In [6], asymptotic error expansion of iterated collocation solution for two-dimensional linear Volterra integral equation was obtained.

In this paper, we extend the two-dimensional transform method (DTM) to solve the two-dimensional nonlinear volterra integral equation and volterra integro differential equation. This technique is based on Taylor series expansion. By using the differential transform method can be transformed the integral equations to algebraic equations, and the resulting algebraic equations are called iterative equations. Also, we apply the three-dimensional transform method (DTM) to solve the three-dimensional nonlinear volterra integral equation. In addition to, we introduce the basic theorems for two and three dimensional differential transform method which we need to solve our equations. Finally, some numerical examples are given in illustration of this method.

2 Two Dimensional Volterra Integral Equation

We will consider the nonlinear Volterra integral equation of the second kind in the following form

$$u(x,y)=g(x,y)+\int_0^x\int_0^y \psi(x,y,t,s,u(t,s))dsdt \qquad (1)$$

Where $u(x,y)$ is an unknown function, the functions $g(x,y)$ and $\psi(x,y,t,s,u)$ are given analytical functions defined, respectively, on $D=[0,X]\times[0,Y]$ and $E=\{(x,y,t,s,u):\ 0\le t\le x\le X, 0\le s\le y\le Y, -\infty\le u\le\infty\}$.

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3 Two Dimensional Volterra Integro-Partial Differential Equation

An area of increasing scientific interest over the past decades is the study of volterra integro-differential equation. This equation is encountered in various applications such as physics, mechanics, and applied science. A general form of the Volterra integro-partial differential equation can be written as

$$u_{xx}+u_{xt}+u_{tt}+u(t,x)=g(t,x)+\int_0^x\int_0^t k(t,s,x,y)F(u(s,y))dyds, \quad (2)$$

where $(t,x)\in[0,A_1]\times[0,A_2]$,

with given supplementary conditions, where $u(t,x)$ is an unknown function which should be determined, $g(t,x)$ and $k(t,s,x,y)$ are analytical functions, respectively.

In this section, we consider the nonlinear function $F(u(s,y))$ in the following form $F(u(s,y)=u^p(s,y)$, where p is a positive integer. With regard to the fact that every finite interval can be transformed to $[0,1]$ by linear map, without loss of generality, we can consider $A_1=A_2=1$.

4 Two dimensional Differential Transform Method

Consider the analytical function of two variable $u(x,y)$, which is defined on $D=[0,X]\times[0,Y]\subseteq \mathbb{R}^2$ and $(x_0,y_0)\in D$. The two-dimensional differential transform of $u(x,y)$ is denoted by $U(k,h)$ and is defined as the following

$$U(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k\partial y^h}\right]_{(x_0,y_0)}, \quad (3)$$

where $u(x,y)$ is the original function, and $U(k,h)$ is called the transformed function.

Inverse differential transform of $U(k,h)$ in definition (3) is defined as follows

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)(x-x_0)^k(y-y_0)^h, \quad (4)$$

By combining def.(3) and (4) with $(x_0,y_0)=(0,0)$, then the function $u(x,y)$ can be written in the form

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k\partial y^h}\right]_{(0,0)}x^ky^h. \quad (5)$$

The fundamental mathematical properties of two-dimensional differential transform method can readily be obtained and are expressed in the following basic theorems.

Theorem 1. *If* $u(x,y)=f(x)g(y)$ *then*, $U(k,h)=F(k)G(h)$.

Theorem 2. *If* $U(k,h),F(k,h)$ *and* $G(k,h)$ *are two dimensional differential transforms of the functions* $u(x,y)$, $f(x,y)$ *and* $g(x,y)$ *respectively, we have*

1. *If* $u(x,y)=f(x,y)\pm g(x,y)$, *then* $U(k,h)=F(k,h)\pm G(k,h)$.
2. *If* $u(x,y)=\alpha f(x,y)$, *then* $U(k,h)=\alpha F(k,h)$.
3. *If* $u(x,y)=f(x,y)g(x,y)$, *then* $U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}F(r,h-s)G(k-r,s)$.
4. *If* $u(x,y)=\frac{\partial f(x,y)}{\partial x}$, *then* $U(k,h)=(k+1)F(k+1,h)$.
5. *If* $u(x,y)=\frac{\partial f(x,y)}{\partial y}$, *then* $U(k,h)=(h+1)F(k,h+1)$.
6. *If* $u(x,y)=\frac{\partial^{r+s}f(x,y)}{\partial^r x\partial^s y}$, *then* $U(k,h)=(k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)F(k+r,h+s)$.
7. *If* $u(x,y)=x^my^n$, *then* $U(k,h)=\delta(k-m,h-n)=\delta(k-m)\delta(h-n)$.
8. *If* $u(x,y)=x^m\sin(at+b)$, *then* $U(k,h)=\frac{a^h}{h!}\delta_{k,m}\sin(\frac{h\pi}{2}+b)$.
9. *If* $u(x,y)=x^m\cos(at+b)$, *then*
10. *If* $u(x,y)=x^me^{at}$, *then* $U(k,h)=\frac{a^h}{h!}\delta_{k,m}$.

Theorem 3. *If* $u(x,y)=\int_0^x\int_0^y f(s,t)dsdt$, $U(k,h)$ *and* $F(k,h)$ *are differential transforms of the functions* $u(x,y)$ *and* $f(x,y)$ *respectively, then*

$$U(k,h)=\begin{cases}0 & \text{,if } k=0 \text{ or } h=0;\\ \frac{F(k-1,h-1)}{kh} & \text{if } k,h=1,2,3,\cdots\end{cases} \quad (6)$$

Theorem 4. *If* $u(x,y)=\int_0^x\int_0^y f(s,t)g(s,t)dsdt$, $U(k,h)$, $F(k,h)$ *and* $G(k,h)$ *are differential transforms of the functions* $u(x,y)$, $f(x,y)$ *and* $g(x,y)$ *, respectively, then*

$$U(k,h)=\begin{cases}0 & \text{,if } k=0 \text{ or } h=0;\\ \frac{1}{kh}\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}F(r,h-s-1)G(k-r-1,s), & \text{if k, h}=1,2,3,\cdots\end{cases} \quad (7)$$

Theorem 5. *Suppose that* $u(x,y)=\prod_{i=1}^{n}f_i(x,y)$. *If* $U(k,h)$ *and* $F_i(k,h)$ *for* $i=1,2,...n$ *are differential transforms of the functions* $u(x,y)$ *and* $f_i(x,y)$ *for* $i=1,2,...n$ *, respectively, then*

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$$U(k,h)=\sum_{r_{n-1}=0}^{k}\sum_{s_{n-1}=0}^{h}\sum_{s_{n-2}=0}^{r_{n-1}}\sum_{s_{n-2}=0}^{s_{n-1}}\cdots\sum_{r_2=0}^{r_3}\sum_{s_2=0}^{s_3}$$
$$\times\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}F_1(r_1,s_1)F_2(r_2-r_1,s_2-s_1)\cdots\cdot \quad (8)$$
$$\times F_{n-1}(r_{n-1}-r_{n-2},s_{n-1}-s_{n-2})$$
$$\times F_n(k-r_{n-1},h-s_{n-1})$$

The next corollary is the direct result of the Theorems 4 and 5.

Corollary 6. *If the assumptions of the theorem mult.fun are satisfied and*

$$u(x,y)=\int_0^x\int_0^y\prod_{i=1}^{n}f_i(s,t)dsdt,$$

then $U(k,h)=0$ *, if* $k=0$ *or* $h=0$ *and*

$$U(k,h)=\frac{1}{kh}\sum_{r_{n-1}=0}^{k}\sum_{s_{n-1}=0}^{h}\sum_{s_{n-2}=0}^{r_{n-1}}\sum_{s_{n-2}=0}^{s_{n-1}}\cdots\sum_{r_2=0}^{r_3}\sum_{s_2=0}^{s_3}$$

$$\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}F_1(r_1,s_1)F_2(r_2-r_1,s_2-s_1)\ldots F_{n-1}(r_{n-1}-r_{n-2},s_{n-1}-s_{n-2}) \quad (9)$$
$$\times F_n(k-r_{n-1}-1,h-s_{n-1}-1)],$$

where $k\geq 1, h\geq 1.$

Theorem 7. *Assume that if* $F(k)$ *,* $G(h)$ *and* $U(k,h)$ *are differential transforms of* $f(x)$*,* $g(x)$ *and* $u(x,y)$*, respectively, so we have*

1. *If* $f(x)=\left.\frac{\partial u_{(x,y)}}{\partial y}\right|_{y=0}$ *, then* $F(k)=U(k,l), k=0,1,2,....$
2. *If* $f(x)=\left.\frac{\partial u_{(x,y)}}{\partial y}\right|_{y=1}$ *, then* $F(k)=\sum_{s=0}^{n}U(k,s), k=0,1,2,...$
3. *If* $f(x)=\left.\frac{\partial u_{(x,y)}}{\partial x}\right|_{y=0}$ *, then*
   $F(k)=(k+1)U(k+1,0), k=0,1,2,...$
4. *If* $f(x)=\left.\frac{\partial u_{(x,y)}}{\partial x}\right|_{y=1}$ *, then*
   $F(k)=(k+1)\sum_{r=0}^{n}U(k+1,r), k=0,1,2,...$
5. *If* $g(y)=\left.\frac{\partial u_{(x,y)}}{\partial y}\right|_{x=0}$ *,then*
   $G(h)=(h+1)U(0,h+1), h=0,1,2,...$
6. *If* $g(y)=\left.\frac{\partial u_{(x,y)}}{\partial y}\right|_{x=1}$ *, then*
   $G(h)=(h+1)\sum_{r=0}^{m}U(r,h+1), h=0,1,2,...$
7. *If* $g(y)=\left.\frac{\partial u_{(x,y)}}{\partial x}\right|_{x=0}$ *, then* $G(h)=U(1,h), h=0,1,2,...$
8. *If* $g(y)=\left.\frac{\partial u_{(x,y)}}{\partial x}\right|_{x=1}$ *, then*
   $G(h)=\sum_{s=0}^{m}sU(s,h), h=0,1,2,...$

5 Error Analysis

In this section, we perform the estimating error for the integral equations. Since the truncated Taylor series or the corresponding polynomial expansion to the nonlinear volterra integral equation defined into eq.(1) as

$$u(x,y)=g(x,y)+\int_0^x\int_0^y\phi(x,y,r,s,u)dsdr \quad (10)$$

and the approximate solution is defined by

$$u_{p,q}(x,y)=\sum_{k=0}^{p}\sum_{h=0}^{q}U(k,h)(x-x_o)^k(y-y_o)^h,$$

we define $e_{p,q}(x,y)$ as the error function in the following form

$$e_{p,q}(x,y)=\left|u(x,y)-u_{p,q}(x,y)\right|,$$

then, we increase p and q as far as the following inequality holds at each point (x,y)

$$e_{p,q}(x,y)\leq 10^{-m}.$$

where m is any positive integer. In other words, by increasing p and q, the error function $e_p(x)$ approaches to zero.

6 Applications and Results

In this section, we will apply the two dimensional differential transform method for solving some numerical examples of two-dimensional volterra integral and integro-differential equations to show the accuracy and efficiency of the DTM . This method help us to obtain exact solutions for some examples and approximate solutions for the other examples.

Example 1. Consider the following Volterra Integro-partial differential equations

$$\frac{\partial u(x,y)}{\partial x}+\frac{\partial u(x,y)}{\partial y}=-1+e^x+e^y+e^{x+y}+\int_0^x\int_0^y u(s,t)dsdt \quad (11)$$

Subject to the initial conditions

$$\begin{cases}u(x,0)=e^x\\ u(0,y)=e^y.\end{cases} \quad (12)$$

Using DTM for eq.(11) and (12), we obtain the following recurrence relation

$$(k+1)U(K+1,h)+(h+1)U(k,h+1)=-\delta(k)\delta(h)$$
$$+\frac{\delta(h)}{k!}+\frac{\delta(k)}{h!}+\frac{1}{k!h!}\frac{U(k-1,h-1)}{kh}, \quad (13)$$

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and the initial conditions will be transformed as follows

$$\begin{cases} U(k,0)=\frac{1}{k!}, k=0,1,2,\ldots \\ U(0,h)=\frac{1}{h!}, h=0,1,2,\ldots \end{cases}, \tag{14}$$

respectively. By substituting into eq.(4), we obtain the closed form of the solution series as follows

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k y^h$$

$$u(x,y)=(1+x+\frac{x^2}{2!}+\ldots)(1+y+\frac{y^2}{2!}+\ldots)=e^{x+y}.$$

which is the exact solution of eq.(11).

Example 2. Consider the following homogeneous nonlinear Gas Dynamic equation [7]

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}-u(1-u)=0, \tag{15}$$

with the initial condition

$$u(x,0)=e^{-x}, \tag{16}$$

whose exact solution can be expressed as

$$u(x,t)=e^{t-x}, \tag{17}$$

Taking two-dimensional differential transform of eq.(15), we obtain the following recurrence relation

$$\begin{aligned}(h+1)U(k,h+1)=&-\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s)\\ &+U(k,h)-\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)U(k-r,s),\end{aligned} \tag{18}$$

the related initial condition should be also transformed as a follows

$$u(k,0)=\frac{(-1)^k}{k!}, K=0,1,2,\ldots, \tag{19}$$

substituting from eq.(19) into eq.(18), and by recursive method, the result is listed as follows

$$U(0,1)=1, \qquad U(1,1)=-1$$

$$U(0,2)=\frac{1}{2!}, \qquad U(2,1)=\frac{1}{2!}$$

$$U(1,2)=-\frac{1}{2!}, \tag{20}$$

substituting all $U(k,h)$ into eq.(4), we obtain the closed form of the solution series as follows

$$u(x,t)=(1-x+t-xt+\frac{t^2}{2!}+\frac{x^2}{2!}+\frac{x^2t}{2!}-\frac{xt^2}{2!}+\ldots)=e^{t-x}.$$

Example 3. Consider the following equation [1]

where $g(x,t)=xe^t-\frac{1}{2}x^4t+\frac{1}{2}x^4te^t$

with initial conditions $u(0,t)=0, \quad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}=e^t$. (23)

Applying DTM for eq.(22) and (23), we get the following recurrence relation

$$(k+1)(k+2)U(k+2,h)+(h+1)(h+2)U(k,h+2)=$$

$$\frac{1}{h!}\delta_{k,1}-\frac{1}{2}\delta_{k,4}\delta_{h,1}+\frac{1}{2}\sum_{k_1=1}^{k}\sum_{k_2=1}^{h}\delta_{k_1,4}\delta_{k_2,1}\delta_{k,k_1}$$

$$-\sum_{k_1=1}^{k}\sum_{k_2=1}^{h}\frac{1}{k_1k_2}\delta_{k-k_1,2}\delta_{h-k_2,1}U(k_1-1,k_2-1)$$

with initial conditions $U(k,0)=\delta_{k,0}, \quad U(1,h)=\frac{1}{h!}$

from initial conditions

then $U(3,h)=0$,

and so on, we continue the same pattern then substitute all $U(k,h)$ into eq.(4),we have series solution as follows

The following error table describes the error from using the block pulse in two dimension method [1] by Aghazadeh, Nasser, and Amir A. K. (2013).

(x,t)	$\lvert e_{5,5}(x,y)\rvert$ [1]	$\lvert e_{6,6}(x,y)\rvert$ [1]	$m=32$	$m=64$
(0.01, 0.01)	1.666×10^{-7}	1.666×10^{-7}	2.238×10^{-8}	2.124×10^{-8}
(0.02, 0.02)	1.333×10^{-6}	1.333×10^{-6}	7.980×10^{-8}	7.742×10^{-8}
(0.1, 0.1)	1.670×10^{-4}	1.666×10^{-4}	6.809×10^{-6}	4.714×10^{-6}
(0.2, 0.2)	1.347×10^{-3}	1.332×10^{-3}	2.334×10^{-4}	2.230×10^{-4}

However the DTM method got the exact solution as shown in eq.(24) without any computational difficulties, which proves the efficiency of the DTM.

Example 4. Consider the following equation [1]

$$u(x,t)+\frac{\partial^2 u(x,t)}{\partial^2 t}=g(x,t)+\int_0^t\int_0^x (y+\cos z)u^2(y,z)dydz, \qquad x,t\in[0,1], \tag{25}$$

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where $g(x,t)=\frac{1}{8}x^4\sin t\cos t-\frac{1}{8}x^4 t-\frac{1}{9}x^3\sin^3 t,$

with intial conditions $u(x,0)=0, \quad \frac{\partial u(x,0)}{\partial t}=x.$

Applying DTM for eq.(25), we get the following recurrence relation

$$U(k,h)+(h+1)(h+2)U(k,h+2)=\frac{1}{16}\frac{1}{h!}\delta_{k,4}2^h\sin\left(\frac{h\pi}{2}\right)-\frac{1}{8}\delta_{k,4}\delta_{h,1}$$

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dimensional differential transform method can readily be obtained and are expressed in the following basic theorems.

Theorem 8. *If* $U(k,h,l)$, $F(k,h,l)$ *and* $G(k,h,l)$ *are three-dimensional differential transform of the functions* $u(x,y,z)$, $f(x,y,z)$ *and* $g(x,y,z)$ *respectively, then*

1. If $u(x,y,z)=f(x,y,z)+g(x,y,z)$ then, $U(k,h,l)=F(k,h,l)+G(k,h,l).$

2. If $u(x,y,z)=x^p y^q z^r$, then $U(k,h,l)=\delta(k-p)\delta(h-q)\delta(l-r).$

3. If $u(x,y,z)=\dfrac{\partial^{p+q+r} f(x,y,z)}{\partial x^p \partial y^q \partial z^l}$, then
$$U(k,h,l)=(k+1)\dots(k+p)(h+1)\dots(h+q)\times(l+1)\dots(l+r)F(k+p,h+q,l+r).$$

4. If $u(x,y,z)=\sin(\alpha x+by+cz)$, then
$$U(k,h,l)=\frac{a^k b^h c^l}{k!h!l!}\sin\left(\frac{k+h+l}{2}\pi\right).$$

5. If $u(x,y,z)=e^{ax+by+cz}$, then $U(k,h,l)=\dfrac{a^k b^h c^l}{k!h!l!}.$

Theorem 9. *If* $u(x,y,z)=f(x,y,z)\ g(x,y,z)$, *then*
$$U(k,h,l)=\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{t=0}^{l}F(r,s,t)G(k-r,h-s,l-t).$$

Theorem 10.

If $u(x,y,z)=f_1(x,y,z)f_2(x,y,z)\dots f_n(x,y,z)$, then
$$U(k,h,l)=\sum_{r_{n-1}=0}^{k}\sum_{r_{n-2}=0}^{r_{n-1}}\dots\sum_{r_1=0}^{r_2}\sum_{s_{n-1}=0}^{h}\sum_{s_{n-2}=0}^{s_{n-1}}\dots\sum_{s_1=0}^{s_2}\sum_{t_{n-1}=0}^{l}\sum_{t_{n-2}=0}^{t_{n-1}}\dots\sum_{t_1}^{t_2}F_1(r_1,s_1,t_1)$$
$$\times F_2(r_2-r_1,s_2-s_1,t_2-t_1)\times\dots\times F_{n-1}(r_{n-1}-r_{n-2},s_{n-1}-s_{n-2},t_{n-1}-t_{n-2})$$
$$\times F_n(k-r_{n-1},h-s_{n-1},l-t_{n-1})$$

Theorem 11.

If $u(x,y,z)=\int_0^x\int_0^y\int_0^z f(r,s,t)dtdsdr$, then
$$U(k,h,l)=\begin{cases}0 & \text{if } k=0 \text{ or } h=0 \text{ or } l=0\\ \frac{1}{khl}F(k-1,h-1,l-1) \text{ if} & k,h,l=1,2,3\end{cases}$$

Theorem 12.

If $u(x,y,z)=f_1(x,y,z)\int_0^x\int_0^y\int_0^z f_2(r,s,t)dtdsdr$, then
$$U(k,h,l)=\sum_{r=1}^{k}\sum_{s=1}^{h}\sum_{t=1}^{l}\left[\tfrac{1}{(k-r,h-s,l-t)}F_1(r,s,t)F_2(k-r-1,h-s-1,l-t-1)\right],$$
where $k=1,2,\dots,p$, $h=1,2,\dots q$, *and* $l=1,2,\dots,n$
$$U(k,h,l)=0 \ \text{if} \ k=0 \ \text{or} \ h=0 \ \text{or} \ l=0$$

9 Error Analysis

In this section, we perform the estimating error for the integral equations. Since the truncated Taylor series or the corresponding polynomial expansion is an approximate solution of the equation
$$u(x,y,z)=g(x,y,z)+\int_0^x\int_0^y\int_0^z\phi(x,y,z,r,s,t,u)dtdsdr.$$
If we define $e_{p,q,n}(x,y,z)$ as an error function in the following form
$$e_{p,q,n}(x,y,z)=\left|u(x,y,z)-u_{p,q,n}(x,y,z)\right|,$$
where
$$u_{p,q,n}(x,y,z)=\sum_{k=0}^{p}\sum_{h=0}^{q}\sum_{l=0}^{n}U(k,h,l)(x-x_o)^k(y-y_o)^h(z-z_o)^l,$$
is approximation function, then we can prescribe $e_{p,q,n}(x,y,z)\le 10^{-m}$, where m is any positive integer, then we increase p and q as far as the following inequality holds at each points (x,y,z)
$$e_{p,q,n}(x,y,z)\le 10^{-m}.$$
In other words, by increasing p,q *and* n, the error function $e_{p,q,n}(x,y,z)$ approaches to zero.

10 Applications and Results

In this section, we will apply the three dimensional differential transform method for solving some numerical examples of three-dimensional volterra integral equations to show the accuracy and efficiency of the DTM.

Example 5. Consider the nonlinear Volterra Integral equation [3]
$$u(x,y,z)=g(x,y,z)-\int_0^x\int_0^y\int_0^z u(r,s,t)dtdsdr, \qquad (34)$$
where $(x,y,z)\in[0,1]\times[0,1]\times[0,1]$
$$g(x,y,z)=\frac{x^2yz+xy^2z+xyz^2}{2}+x+y+z.$$
Eq.(34) has exact solution can be expressed as

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$u(x,y,z) = x + y + z.$

We set

$$f(x,y,z) = \int_0^x \int_0^y \int_0^z u(r,s,t)\,dt\,ds\,dr$$

Then we have $u(x,y,z) = g(x,y,z) - f(x,y,z),$ (36)

taking the DTM of this equation , we obtain

$U(k,h,l) = G(k,h,l) - F(k,h,l),$

where

$$\begin{aligned} G(k,h,l) = \frac{1}{2}[&\delta(k-2)\delta(h-1)\delta(l-1) + \delta(k-1)\delta(h-2)\delta(l-1) \\ &+\delta(k-1)\delta(h-1)\delta(l-2)] + \delta(k-1)\delta(h)\delta(l) \\ &+\delta(k)\delta(h-1)\delta(l) + \delta(k)\delta(h)\delta(l-1), \end{aligned} \quad (37)$$

for $k = 0,1,\ \ ,p, h = 0,1,\ \ ,q$ and $l = 0,1,\ \ ,n$ and

$$F(k,h,l) = \frac{1}{khl} U(k-1,h-1,l-1), k \geq 1, h \geq 1, l \geq 1, \quad (38)$$

for $p = 1, q = 2, l = 3$, and recursive method, we obtain

now, by substituting this relation in equation (32), we obtain the closed form of the solution series as follows

$$u(x,y,z) = x + y + z. \quad (39)$$

Example 6. Consider nonlinear Volterra Integral equation [3]

$$u(x,y,z) = g(x,y,z) - 24x^2 y \int_0^x \int_0^y \int_0^z u(r,s,t)\,dt\,ds\,dr \quad (40)$$

where $(x,y,z) \in [0,1] \times [0,1] \times [0,1]$ and

Eq. (40) has exact solution can be expressed as

$u(x,y,z) = x^2 y + yz^2 + xyz.$

We set

$$f(x,y,z) = -24x^2 y \int_0^x \int_0^y \int_0^z u(r,s,t)\,dt\,ds\,dr$$

then, we have

$u(x,y,z) = g(x,y,z) + f(x,y,z).$

If we take the DTM of this equation, we obtain

$U(k,h,l) = G(k,h,l) + F(k,h,l)$

where

$$\begin{aligned} G(k,h,l) = &4\delta(k-5)\delta(h-3)\delta(l-1) + 4\delta(k-3)\delta(h-3)\delta(l-3) \\ &+ 4\delta(k-4)\delta(h-3)\delta(l-2) \\ &+\delta(k-2)\delta(h-1)\delta(l) + \delta(k)\delta(h-1)\delta(l-2) \\ &+ \delta(k-1)\delta(h-1)\delta(l-1) \end{aligned}$$

for $k = 0,1,\ \ ,p, h = 0,1,\ \ ,q$ and $l = 0,1,\ \ ,n$ and

$$F(k,h,l) = -24 \sum_{r=0}^{p} \sum_{s=0}^{q} \sum_{t=0}^{n} \frac{1}{(k-r)(h-s)(l-t)} \delta(k-2)\delta(h-1)\delta(l) \times U(k-r-1, h-s-1, l-t-1)$$

for $k = 1,\ \ ,p, h = 1,\ \ ,q$ and $l = 1,\ \ ,n$, by solving the above recursive equations for $p = 3, q = 1$ and $n = 3$, the result are listed as follows

now, by substituting this relation in equation (32), we get $u(x,y,z) = x^2 y + yz^2 + xyz$, which is the exact solution of eq.(40).

Example 7. Consider nonlinear Volterra Integral [3]

$$u(x,y,z) = g(x,y,z) + \int_0^x \int_0^y \int_0^z u(r,s,t)\,dt\,ds\,dr, \quad (41)$$

where $(x,y,z) \in [0,1] \times [0,1] \times [0,1]$ and

$$g(x,y,z) = e^{x+y} + e^{x+z} + e^{y+z} - e^x - e^y - e^z + 1.$$

Eq. (41) has the exact solution can be expressed as

$u(x,y,z) = e^{x+y+z}$.

In the same manner as in the previous example, we set

$$f(x,y,z) = \int_0^x \int_0^y \int_0^z u(r,s,t)\,dt\,ds\,dr,$$

then, we have

$u(x,y,z) = g(x,y,z) + f(x,y,z),$

taking the differential transformation of this equation, we obtain

$U(k,h,l) = G(k,h,l) + F(k,h,l),$

where

$$G(k,h,0) = \frac{1}{k!h!}, k = 0,1,\ldots p, h = 0,1,\ldots, q$$

$$G(0,h,l) = \frac{1}{h!l!}, l = 0,1,\ldots n, h = 0,1,\ldots, q$$

$$G(k,0,l) = \frac{1}{k!l!}, k = 0,1,\ldots p, l = 0,1,\ldots, n$$

$$G(k,h,l) = 0, k,h,l \neq 0$$

and we also have

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$$F(k,h,l)=\frac{1}{khl}U(k-1,h-1,l-1)$$

where $F(k,h,0)=F(k,0,l)=F(0,h,l)=0$, $k=0,1,\ldots p, h=0,1,\ldots,q, l=0,1,\ldots,n,$ by solving the above recursive equations for $p=q=n=2$ and $p=q=n=3$, we obtain

$$\begin{aligned} u_{2,2,2}(x,y,z) &= 1+(x+y+z)+\frac{1}{2}(x+y+z)^2 \\ &+\frac{1}{2}(x^2y+xy^2+x^2z+xz^2+y^2z+yz^2) \\ &+\frac{1}{4}(x^2y^2+x^2z^2+y^2z^2)+\frac{1}{2}(xyz^2+xy^2z+x^2yz) \\ &+\frac{1}{4}(xy^2z^2+x^2yz^2+x^2y^2z)+\frac{1}{8}x^2y^2z^2, \end{aligned}$$

$$\begin{aligned} u_{3,3,3}(x,y,z) &= 1+(x+y+z)+\frac{1}{2}(x+y+z)^2+\frac{1}{6}(x+y+z)^3 \\ &+\frac{1}{4}(x^2y^2+x^2z^2+y^2z^2) \\ &+\frac{1}{6}(x^3y+xy^3+x^3z+xz^3+y^3z+yz^3) \\ &+\frac{1}{12}(x^2y^3+x^3y^2+x^2z^3+x^3z^2+y^2z^3+y^3z^2) \\ &+\frac{1}{36}(x^3y^3+x^3z^3+y^3z^3) \\ &+\frac{1}{2}(xyz^2+xy^2z+x^2yz)+\frac{1}{4}(xy^2z^2+x^2yz^2+x^2y^2z) \\ &+\frac{1}{6}(xyz^3+xy^3z+x^3yz)+\frac{1}{12}(xy^2z^3+xy^3z^2+x^2yz^3+x^3yz^2+x^2y^3z \\ &+x^3y^2z)+\frac{1}{36}(xy^3z^3+x^3y^3z)+\frac{1}{24}(x^2y^2z^3+x^2y^3z^2+x^3y^2z^2) \\ &+\frac{1}{72}(x^2y^3z^3+x^3y^2z^3+x^3y^3z^2)+\frac{1}{8}x^2y^2z^2++\frac{1}{216}x^3y^3z^3, \end{aligned}$$

which are truncated Taylor series of exact solution, the following table shows the absolute errors at some particular points

x	y	z	$ExactSolution$	$e_{2,2,2}(x,y,z)$	$u_{3,3,3}(x,y,z)$
0.1	0.1	0.1	1.3498588076	1.6261825×10^{-3}	1.557798×10^{-5}
0.01	0.01	0.01	1.23367880600	4.8175870×10^{-4}	9.492003×10^{-6}
0.01	0.01	0.01	1.1274968516	1.8474381×10^{-4}	4.338226×10^{-6}
0.01	0.01	0.01	1.0304545340	1.5113801×10^{-6}	1.280568×10^{-9}
0.001	0.001	0.001	1.0212220516	4.3803265×10^{-7}	8.421920×10^{-10}
0.001	0.001	0.001	1.0120722889	1.7775346×10^{-7}	4.165485×10^{-10}
0.001	0.001	0.001	1.0030045045	1.5043042×10^{-9}	3.637978×10^{-12}

According to the table , these results indicate that the use of time steps smaller than about 0.1, the error function approaches to zero.

11 CONCLUSIONS

In this study, we introduced the definition and operation of two and three-dimensional differential transform. Integral equations can be transformed to algebraic equations by using the differential transform and the resulting algebraic equations are called iterative equations. The overall spectra can be calculated through the initial condition in association with the iterative equations. Finally, by using this algebraic equations, we find the approximate solution of the integral equations.

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